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Friction factors for idealised 2D stratified channel flow

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ABSTRACT

Friction factors and associated friction velocities are important flow characteristics. Analytical expressions of wall and interface friction properties are derived for idealised three layered and two layered pressure driven stratified flow in a two-dimensional channel under the assumption of fully developed laminar viscous flow. Channel inclination is accounted for as well. The influence of flow and geometrical channel parameters is then concretely assessed for water/air and oil/air interfaces due to their potential relevance to two phase flows occurring in the human airways.

1. Introduction

Liquid/gas flow plays a crucial role in industry and is therefore well studied in this context. Analytical flow models applied to predict flow through the human airways on the other hand commonly make the assumption of single phase flow and therefore neglect the potential influence of a liquid layer adjacent to the walls on flow induced phenomena. A particular example of such a phenomenon is human voiced speech sound or phonation for which physical models use single phase flow, *e.g.* [1,2], the same way as was done since pioneering work in the seventies [3]. Nevertheless, besides the obvious presence of air flow, two liquids are of immediate interest for phonation: (1) water since (de-)hydration is reported pertinent [4] and (2) oil since the viscosity ratio approximates values reported for mucus [5]. In this work, a simple analytical two-dimensional (2D) stratified flow model is proposed which allows to take into account the presence of liquid layers adjacent to the channel walls while maintaining major flow assumptions applied in the context of physical phonation modelling, *i.e.* pressure driven laminar steady incompressible 2D flow [1,2]. This way the model accounts in general for liquid/gas/liquid flow and is therefore more general than models presented in literature for gas/liquid flow [6]. Consequently, the proposed model extends potential applications to biological flows such as blood flow as well. An overview of a general inclined channel flow configuration is depicted in Fig. 1 illustrating three fluid layers with thickness h (lower liquid layer), $H-h-h_2$ (gas layer) and h_2 (upper liquid layer) where H denotes the total channel height and β indicates the channel inclination, *i.e.* $0^\circ \leq \beta \leq 90^\circ$ between horizontal ($\beta = 0^\circ$) and vertical ($\beta = 90^\circ$) channel. Analytical expressions of wall and interfacial friction factors and velocities are formulated (Section 2.2) and analysed as a function of fluids, flow and geometrical properties (Section 3). Keeping in mind potential application to phonation or other phenomena related to flow through the upper airways, water/air and oil/air flow configurations are considered and liquid layer thickness is varied from thin to thick. A conclusion is formulated in Section 4.

2. Stratified 2D idealised channel flow

2.1. Model formulation

Steady pressure driven laminar fully developed stratified flow through a uniform 2D channel is considered for one up to three layers of immiscible fluids (subindex $i = \{L, G, L_2\}$) subjected to gravity as schematically depicted in Fig. 1. Interfaces between fluids with dynamic viscosity μ_i and density ρ_i are assumed smooth. Under these assumptions the momentum equation associated with the flow in each layer yields the following set of second order partial differential equations where u_L, u_G and u_{L_2} indicate the velocity profile in each layer respectively and dp/dx denotes the driving pressure gradient:

$$\mu_L \frac{\partial^2 u_L}{\partial y^2} = \frac{dp}{dx} - \rho_L g \sin(\beta); \quad -h \leq y < 0, \tag{1a}$$

$$\mu_G \frac{\partial^2 u_G}{\partial y^2} = \frac{dp}{dx} - \rho_G g \sin(\beta); \quad 0 < y \leq H-h-h_2, \tag{1b}$$

$$\mu_{L_2} \frac{\partial^2 u_{L_2}}{\partial y^2} = \frac{dp}{dx} - \rho_{L_2} g \sin(\beta); \quad H-h-h_2 < y \leq H-h \tag{1c}$$

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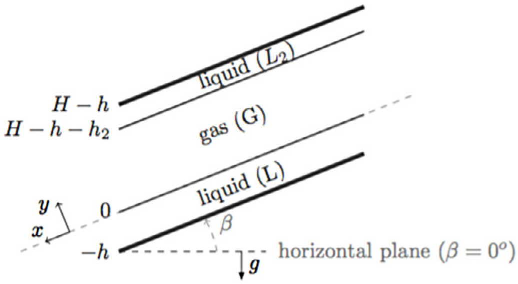


Fig. 1. Schematic overview of the stratified flow configuration: x-direction parallel to rigid channel walls, orthogonal y-axis, channel inclination angle $0 \leq \beta \leq 90°$, total channel height H. Gas (G) layer enveloped by two liquid layers (L, L_2) of height h and h_2 respectively. The y-coordinates associated with outer channel walls ($y = -h$ and $y = H - h$) and smooth liquid/gas interfaces ($y = 0$ and $y = H - h - h_2$) are indicated as is gravitational acceleration g.

subjected to no-slip wall boundary conditions along the lower ($y = -h$) and upper ($y = H - h$) interior wall

$$u_L|_{y=-h} = 0, \tag{2a}$$

$$u_{L_2}|_{y=H-h} = 0, \tag{2b}$$

and interface boundary conditions expressing equal velocity and shear along the lower ($y = 0$) and upper ($y = H-h-h_2$) liquid/gas interface:

$$u_L|_{y=0} = u_G|_{y=0}, \tag{3a}$$

$$u_G|_{y=H-h-h_2} = u_{L_2}|_{y=H-h-h_2}, \tag{3b}$$

$$\mu_L \left.\frac{\partial u_L}{\partial y}\right|_{y=0} = \mu_G \left.\frac{\partial u_G}{\partial y}\right|_{y=0}, \tag{3c}$$

$$\mu_G \left.\frac{\partial u_G}{\partial y}\right|_{y=H-h-h_2} = \mu_{L_2} \left.\frac{\partial u_{L_2}}{\partial y}\right|_{y=H-h-h_2}. \tag{3d}$$

The fully developed laminar velocity profiles (u_L, u_G, u_{L_2}) in each layer are then obtained by integrating momentum Eq. (1). Analytical expressions for the velocity profiles in each layer under the assumption that fluid properties in the liquid outer layers are identical ($\mu_{L_2} = \mu_L$ and $\rho_{L_2} = \rho_L$) are given for general three layer flow, *i.e.* layer thicknesses h (lower liquid layer), $H - h - h_2$ (gas layer) and h_2 (upper liquid layer). Simplified expressions are obtained for symmetrical liquid layers ($h = h_2$), two layer flow ($h_2 = 0$) and well-known [7–9] Poiseuille flow expressions for fully developed single phase laminar flow ($h = h_2 = 0$).

Analytical expressions of mean velocity $\overline{U}$ for each fluid layer follow then from integration of the velocity profiles:

$$\overline{U}_L = \frac{1}{h}\int_{-h}^{0} u_L(y)dy, \tag{4a}$$

$$\overline{U}_G = \frac{1}{H-h-h_2}\int_{0}^{H-h-h_2} u_G(y)dy, \tag{4b}$$

$$\overline{U}_{L_2} = \frac{1}{h_2}\int_{H-h-h_2}^{H-h} u_{L_2}(y)dy. \tag{4c}$$

2.2. Wall and interfacial friction

Wall (subscript w) and interfacial (subscript I) friction are characterised by the shear stress $\tau_{w,I}$ and resulting expressions of friction factor $f_{w,I}$ and friction velocity $v^*_{w,I}$. Analytical expressions of normalised wall and interfacial friction factors are derived as

$$f_{w,I} = \frac{2\tau_{w,I}}{\rho\overline{U}^2}, \tag{5}$$

Table 1
Fluid properties[a].

Fluid	μ [N s/m²]	ρ [kg/m³]
Oil	2.1×10^{-2}	8.6×10^{2}
Water	10^{-3}	10^{3}
Air	1.8×10^{-5}	1.2

[a] temperature $T = 20\,°C$.

with $\overline{U}$ and ρ associated with the pertinent layer adjacent to the wall and $\tau_{w,I}$ indicating wall (subscript w) or interfacial (subscript I) shear stress

$$\tau_{w,I} = \mu \left.\frac{\partial u}{\partial y}\right|_{y_{w,I}} \tag{6}$$

evaluated on either the channel walls or fluids interfaces. Wall and interfacial friction velocity $v^*_{w,I}$ follows then as

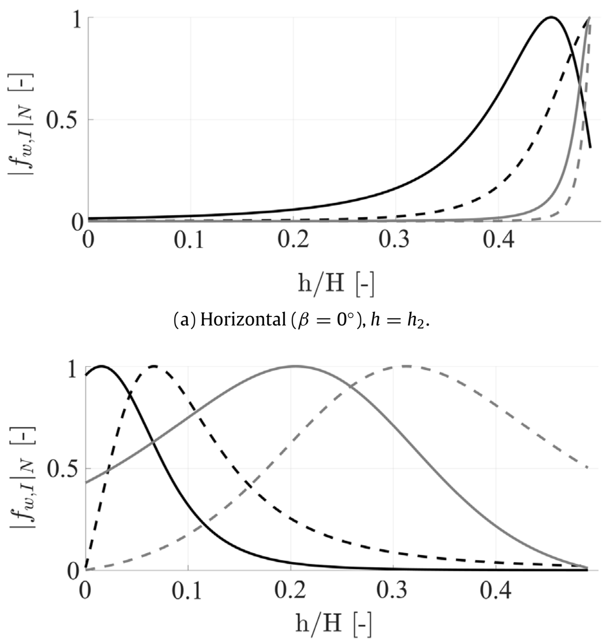
$$v^*_{w,I} = \sqrt{\frac{\tau_{w,I}}{\rho}}. \tag{7}$$

In the case of single phase flow ($h = h_2 = 0$), well known [7–9] expressions of f_w, τ_w and v^*_w associated with Poiseuille flow are obtained as shown in Appendix A. Expressions derived for two layered and three layered stratified flow are given in Appendices B and C, respectively. Superscripts l and u are used to indicate lower (l) and upper (u) wall/interfaces. It is verified that general expressions given in Appendices B and C reduces to expressions for single phase flow as $h_2 = h = 0$.

3. Influence of fluid, channel and flow on friction

Expressions for wall and interfacial friction factors $f_{w,I}$ and related quantities such as friction velocities $v^*_{w,I}$, presented in Section 2.2 and Appendices A–C, can be evaluated as a function of fluid properties ($\mu_{L,G}$, $\rho_{L,G}$ while assuming a single liquid $L = L_2$), geometrical channel properties (H, β) and flow properties (h, h_2, dp/dx). Concretely, in this section, the influence of these quantities on wall and interfacial friction factors is assessed. As outlined in the introduction, three fluids are considered, *i.e.* oil, water and air, for which properties are summarised in Table 1. The following liquid/gas/liquid flow configurations are then studied: water/air/water and oil/air/oil.

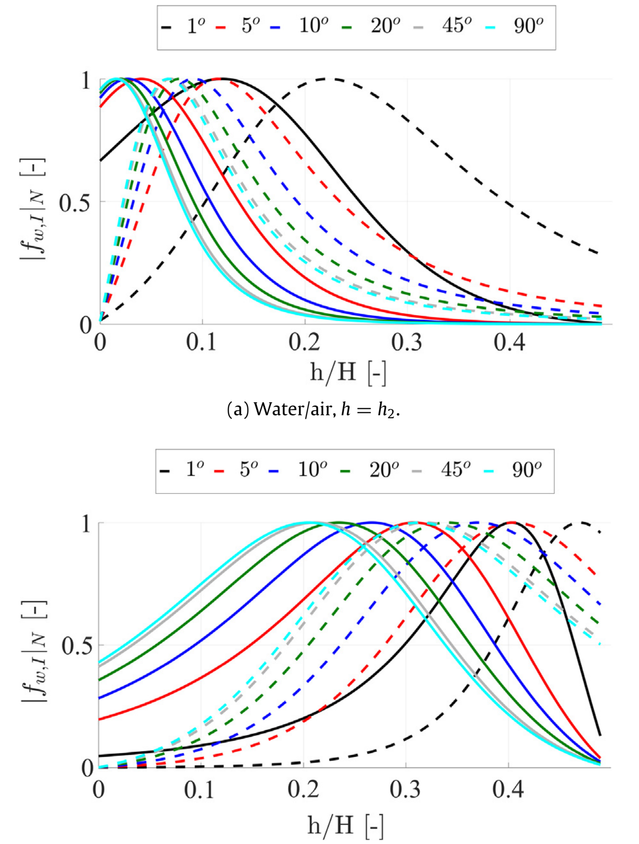
At first, stratified flow configurations with symmetrical liquid layers ($h = h_2$) are considered. From Eq. (14) is seen that for symmetrical liquid layers upper and lower values of wall and interfacial friction factors have equal magnitude ($|f^l_w| = |f^u_w|$ and $|f^l_I| = |f^u_I|$) so that subscripts l and u can be omitted, *i.e.* wall friction factor magnitude $|f_w|$ and interfacial friction factor magnitude $|f_I|$. Fig. 2 illustrates $|f_{w,I}|$ when liquid layer thickness $h/H = h_2/H$ is varied for both water/air (dark curves) and oil/air (grey shaded curves) interfaces in a horizontal (Fig. 2(a), $\beta = 0°$) and vertical (Fig. 2(b), $\beta = 90°$) channel. Note that friction factor magnitudes $|f_{w,I}|$ are normalised by the maximum over the parameter range, *i.e.* $|f_{w,I}|_N = |f_{w,I}|/\max(|f_{w,I}|)$. It is seen that both $|f_w|$ (dashed line) as $|f_I|$ (full line) exhibits a single maximum $\max(|f_{w,I}|)$. In Fig. 2(b), liquid layer thickness h/H associated with $\max(|f_{w,I}|)$ is observed to occur for much thinner liquid layers when oil is replaced by water, *e.g.* for $|f_I|$ the maximum is observed at $h/H \approx 0.2$ for oil/air (grey) and at $h/H \approx 0.02$ for water/air (black). From Fig. 2 (horizontal versus vertical) follows also that the liquid layer thickness associated with $\max(|f_{w,I}|)$ depends on the channel's inclination $0° \leq \beta \leq 90°$. This is shown in more detail in Fig. 3 where the channel's inclination is gradually varied. It is observed that increasing the inclination angle results in decreasing h/H associated with $\max(|f_{w,I}|)$ at a rate imposed by the factor $\sin(\beta)$



(a) Horizontal ($\beta = 0°$), $h = h_2$.

(b) Vertical ($\beta = 90°$), $h = h_2$.

Fig. 2. Normalised wall (dashed lines) and interface (full lines) friction factor magnitudes $|f_{w,I}|_N = |f_{w,I}|/\max(|f_{w,I}|)$ for H = 0.25 cm and $\frac{dp}{dx} = -0.2$ cPa as a function of liquid layer thickness $0.001 \leq h/H \leq 0.49$ for symmetrical ($h = h_2$) stratified water/air (black) and oil/air (grey) flow: (a) Horizontal channel ($\beta = 0°$), (b) Vertical channel ($\beta = 90°$).



(a) Water/air, $h = h_2$.

(b) Oil/air, $h = h_2$.

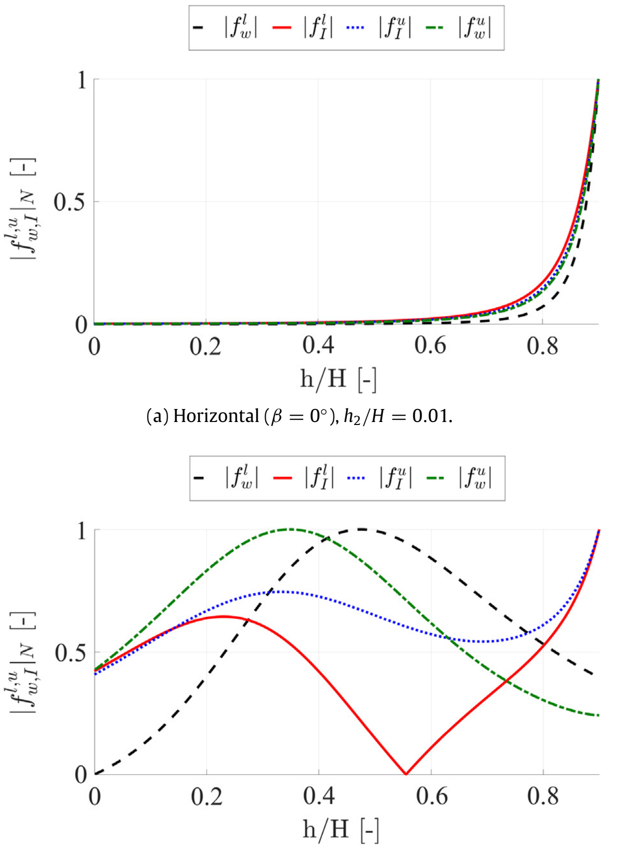
Fig. 3. Normalised wall (dashed lines) and interface (full lines) friction factor magnitudes $|f_{w,I}|_N = |f_{w,I}|/\max(|f_{w,I}|)$ for H = 0.25 cm and $\frac{dp}{dx} = -0.2$ cPa as a function of layer thickness $0.001 \leq h/H \leq 0.49$ for symmetrical ($h = h_2$) stratified flow for different inclination angles β: (a) Water/air, (b) Oil/air.

(see Eq. (14)). Consequently, the decrease of h/H with increasing β is more significant for small then for large inclination angles. Indeed, in Fig. 3 is seen that liquid layer thickness h/H associated with $\max(|f_{w,I}|)$ is approximately unaltered for $\beta \geq 45°$ whereas it varies when the inclination angle is shifted in the range $0° \leq \beta \leq 10°$.

Next, general stratified flow configurations with symmetrical ($h = h_2$) or asymmetrical ($h \neq h_2$) liquid layers are considered. From Eq. (13) is seen that, in general, upper and lower values of wall and interfacial friction factors do not have equal magnitude ($|f_w^l| \neq |f_w^u|$ and $|f_I^l| \neq |f_I^u|$) so that subscripts l and u are not omitted. Fig. 4 illustrates wall and interfacial friction factor magnitudes for stratified oil/air flow for varying lower liquid layer thickness h/H, constant upper layer liquid thickness $h_2/H = 0.01$ and horizontal ($\beta = 0°$, Fig. 4(a)) or vertical ($\beta = 90°$, Fig. 4(b)) channel. For a vertical channel (Fig. 4(b)), it is seen that liquid layer thickness h/H associated with maximum upper and lower wall or interface friction can differ considerably. Plotted wall friction curves (dashed and dashed–dotted lines) exhibit a single maximum as was the case for symmetrical liquid layers. Interfacial friction curves (full and dotted lines) exhibit two local maxima following asymmetry in velocity profile. Consequently, the transverse position of maximum velocity might deflect from channel's centre $H/2$, its position for symmetrical liquid layers.

As for symmetrical configurations, the influence of liquid layer thickness (h/H and h_2/H) on wall and interfacial friction factors is considered for different liquid fluids and channel inclination angles β. Fig. 5 illustrates normalised lower wall (full curves) and interfacial (dashed curves) friction factor magnitudes $|f_{w,I}^l|_N$ for $0 \leq h/H \leq 0.9$ while $H = 0.25$ cm and $h_2/H = 0.01$ is held constant for oil/air (grey curves) and water/air (black curves) stratified flow in a horizontal ($\beta = 0°$, Fig. 5(a)) and vertical ($\beta = 90°$, Fig. 5(b)) channel. The influence of channel inclination β on $|f_{w,I}^l|_N$ is further illustrated in Fig. 6 for the same water/air and oil/air flow configuration, *i.e.* $h_2/H = 0.01$ and $0 \leq h/H \leq 0.9$. It is seen that general tendencies previously observed for symmetrical ($h = h_2$) liquid layers hold for asymmetrical ($h \neq h_2$) liquid layers as well since the position of extrema shifts towards thinner liquid layers (h/H decreases) as the liquid is changed from oil to water and as the inclination angle β increases. From Eq. (13) is seen that the change with β is again governed by $\sin(\beta)$ so that a variation of β is significant for small inclination angles ($0° \leq \beta \leq 10°$) and almost negligible for large inclination angles ($\beta \geq 45°$).

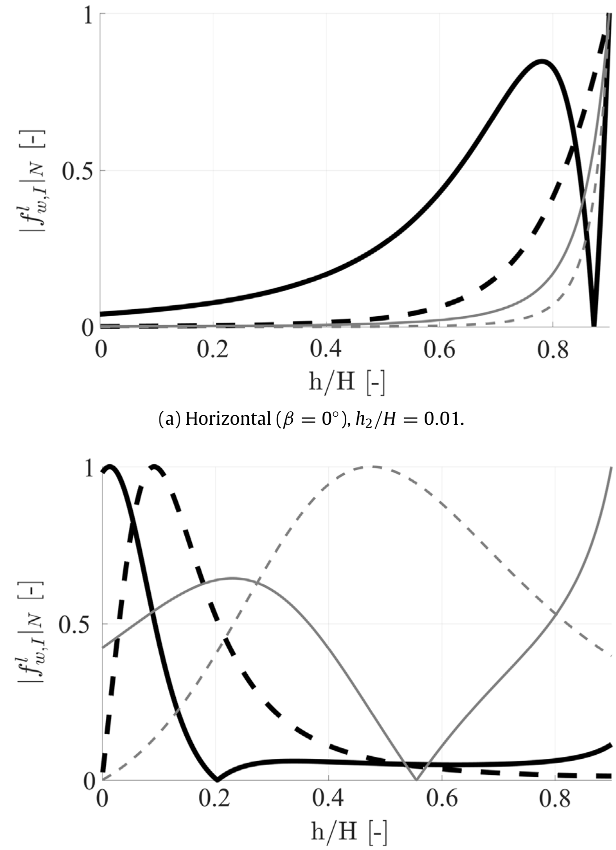
Lower (subscript l) and upper (subscript u) friction factors are further illustrated in Fig. 7 for constant driving pressure gradient, constant amount of liquid and varying total channel height H, $(h + h_2) < H \leq H_m$ with $H_m = 0.25$ cm, for stratified oil/air flow in a vertical channel ($\beta = 90°$). Different liquid configurations are considered, *i.e.* upper liquid layer thickness yields $h_2/H_m = 0.01$ and different values for a constant lower layer thickness are assessed, *i.e.* $h/H_m \in \{0.01, 0.02, 0.3\}$. Consequently, increasing channel height H corresponds to increasing gas layer height $H - h - h_2$. As a tendency, it is seen that $|f^{l,u}|_N$ is low for thick gas layers (low $(h + h_2)/H$ ratios) and that $|f^{l,u}|_N$ is high for thin gas layers (large $(h + h_2)/H$ ratios). For intermediate amounts of liquid (intermediate $(h + h_2)/H$ ratios), $|f^{l,u}|_N$ depends strongly (up to a factor 2) on the liquid layer thickness. Comparing Figs. 7(a) and (b) shows that for thin liquid layers the wall friction $|f_w^{l,u}|_N$ is not much affected since for $h/H_m = 0.02$ the curves are almost similar as to the ones obtained in the symmetrical case $h/H_m = 0.01$



(a) Horizontal ($\beta = 0°$), $h_2/H = 0.01$.

(b) Vertical ($\beta = 90°$), $h_2/H = 0.01$.

Fig. 4. Normalised lower (subscript l) and upper (subscript u) wall (dashed $|f_w^l|$ and dashed–dotted $|f_w^u|$) and interface (full $|f_I^l|$ and dotted $|f_I^u|$) friction factor magnitudes $|f_{w,I}^{l,u}|_N = |f_{w,I}^{l,u}|/\max(|f_{w,I}^{l,u}|)$ for $H = 0.25$ cm and $\frac{dp}{dx} = -0.2$ cPa as a function of liquid layer thickness $0.001 \leq h/H \leq 0.9$ for asymmetrical ($h_2/H = 0.01$) stratified oil/air flow: (a) Horizontal channel ($\beta = 0°$), (b) Vertical channel ($\beta = 90°$).



(a) Horizontal ($\beta = 0°$), $h_2/H = 0.01$.

(b) Vertical ($\beta = 90°$), $h_2/H = 0.01$.

Fig. 5. Normalised lower wall (dashed lines) and interface (full lines) friction factor magnitudes $|f_{w,I}^l|_N = |f_{w,I}^l|/\max(|f_{w,I}^l|)$ for $H = 0.25$ cm and $\frac{dp}{dx} = -0.2$ cPa as a function of liquid layer thickness $0.001 \leq h/H \leq 0.9$ for asymmetrical ($h_2/H = 0.01$) stratified water/air (thick black) and oil/air (thin grey) flow: (a) Horizontal channel ($\beta = 0°$), (b) Vertical channel ($\beta = 90°$).

whereas the interfacial friction $|f_I^{l,u}|_N$ is altered significantly. For thicker liquid layers such as $h/H_m = 0.3$ in Fig. 7(c) both wall $|f_w^{l,u}|_N$ and interfacial $|f_I^{l,u}|_N$ friction are altered significantly with respect to the symmetrical case.

Presented analytical expressions have the advantage that a functional analysis can be applied in order to further search the potential influence of model parameters. Fig. 8 illustrates an analysis with respect to the normalised lower liquid thickness h/H. h/H values associated with the first (smallest h/H) local maximum of normalised interfacial friction factor magnitude ($|f_I^l|$, see *e.g.*, Fig. 6 are plotted as a function of inclination angle β (Fig. 8(a)), normalised upper liquid layer thickness $\alpha_2 = h_2/h$ (Fig. 8(b)) and driving pressure variation $\alpha_p = (\frac{dp}{dx})/(\frac{dp}{dx})_0$ with $(\frac{dp}{dx})_0 = -20$ cPa. In each case, the maximal normalised lower liquid thickness, *i.e.* $\frac{h_m}{H} = \frac{1}{1+\alpha_2}$, is shown as well. Indeed, h_m/H not only indicates the value for which the channel is completely filled with liquid, but also corresponds to the second local maximum observed for asymmetrical liquid layers. Tendencies are easily observed. Fig. 8(a) shows that maximum friction is obtained for thinner lower liquid layers (h/H decreases) as the inclination angle β is increased for both liquids (oil, water). From Fig. 8(c) follows that when α_p, and hence the driving pressure difference, increases maximum friction is obtained for thicker liquid layers (h/H increases) for both liquids (oil, water). Fig. 8(b) shows that liquid properties can influence the general tendencies since when liquid layer ratio $\alpha_2 = \frac{h_2}{h}$ increases, maximum friction is observed for thinner lower liquid layers (h/H decreases) when oil is used. On the other hand, when water is used, h/H corresponding to maximum friction remains almost constant.

4. Conclusion

Analytical expressions of wall and interfacial friction factors are derived for a simplified analytical model of idealised laminar fully developed viscous liquid/gas/liquid stratified 2D channel flow. Analytical functional expressions can then be analysed so that the influence of each of these parameters can be systematically studied. This is illustrated by considering the lower liquid layer thickness associated with maximal interfacial friction. The influence of flow (driving pressure) and geometrical (layer thickness, channel inclination) model parameters is shown for oil/air/oil and water/air/water flow keeping in mind potential future application to human airways. Derived expressions are of particular interest for future application to biological flows.

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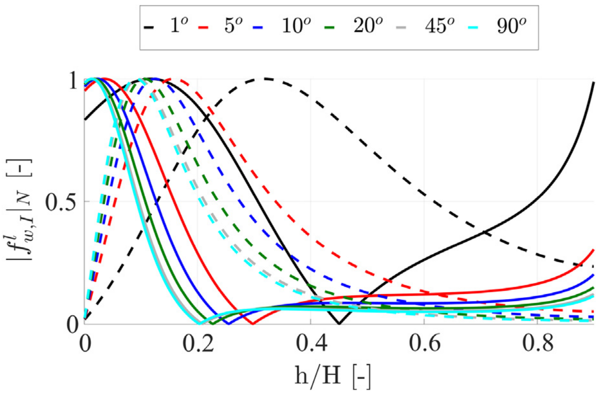
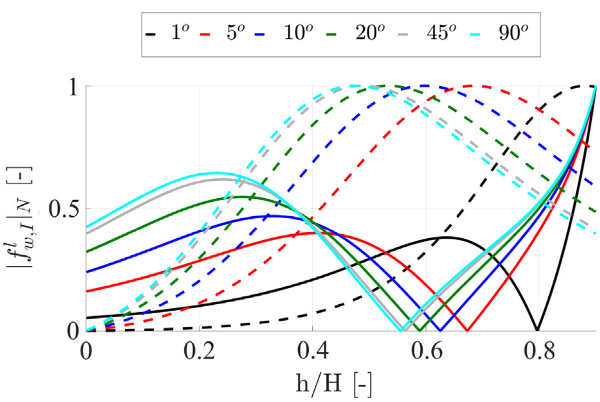
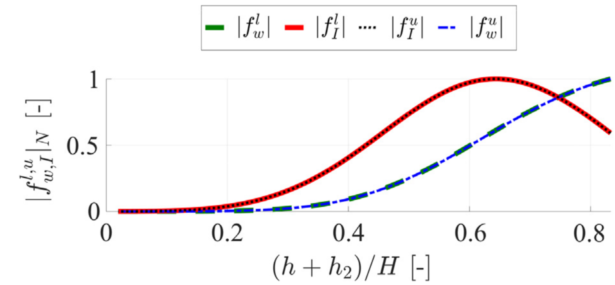


(a) Water/air, $h_2/H = 0.01$.

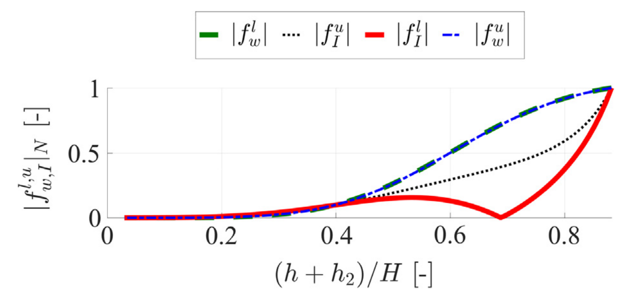


(b) Oil/air, $h_2/H = 0.01$.

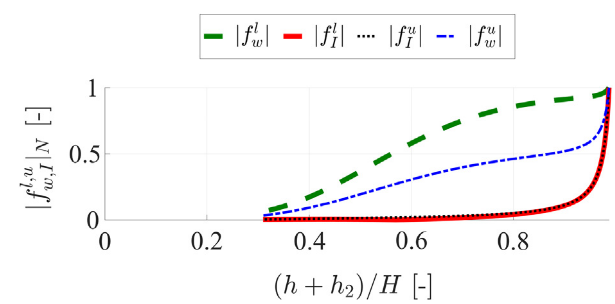
Fig. 6. Normalised lower wall (dashed lines) and interface (full lines) friction factor magnitudes $|f^l_{w,I}|_N = |f^l_{w,I}|/\max(|f^l_{w,I}|)$ for $H = 0.25$ cm and $\frac{dp}{dx} = -0.2$ cPa as a function of layer thickness $0.001 \le h/H \le 0.9$ for asymmetrical ($h_2/H = 0.01$) stratified flow for different inclination angles β: (a) Water/air, (b) Oil/air.



(a) $h/H_m = 0.01$ and $h_2/H_m = 0.01$.



(b) $h/H_m = 0.02$ and $h_2/H_m = 0.01$.



(c) $h/H_m = 0.3$ and $h_2/H_m = 0.01$.

Fig. 7. Normalised lower (subscript l) and upper (subscript u) wall (dashed $|f^l_w|$ and dashed–dotted $|f^u_w|$) and interface (full $|f^l_I|$ and dotted $|f^u_I|$) friction factor magnitudes $|f^{l,u}_{w,I}|_N = |f^{l,u}_{w,I}|/\max(|f^{l,u}_{w,I}|)$ for $\frac{dp}{dx} = -0.2$ cPa as a function of varying channel height $(h+h_2)/H_m \le (h+h_2)/H < 1$ with $H_m = 0.25$ cm for stratified oil/air flow in a vertical channel ($\beta = 90°$) for symmetrical (a) and asymmetrical (b,c) liquid layers.

Appendix A. Single phase flow

For single layer flow ($h = h_2 = 0$ and denoting $\rho = \rho_G$, $\mu = \mu_G$), the mean velocity is given as

$$\overline{U} = \frac{H^2 g\rho \sin(\beta)}{12\mu} - \frac{H^2 \frac{dp}{dx}}{12\mu}. \tag{8}$$

The wall shear stress (Eq. (6)) is then

$$\tau_w = \frac{6\mu\overline{U}}{H} \tag{9}$$

so that wall friction $f_w = f^l_w$ for a single fluid layer ($h = h_2 = 0$) yields (Eq. (5))

$$f_w = \frac{12\mu}{\rho H\overline{U}} \qquad \text{or} \qquad f_w = \frac{12}{Re_H} \tag{10}$$

with $Re_H = \frac{\rho H\overline{U}}{\mu}$. Note that $f^u_w = -f^l_w$ holds for a single fluid layer. The wall friction velocity (Eq. (7)) becomes

$$v^*_w = \sqrt{6}\sqrt{\frac{\mu\overline{U}}{\rho H}}. \tag{11}$$

For a horizontal channel ($\beta = 0$), so that $\sin\beta = 0$ and the first term in Eq. (8) drops out.

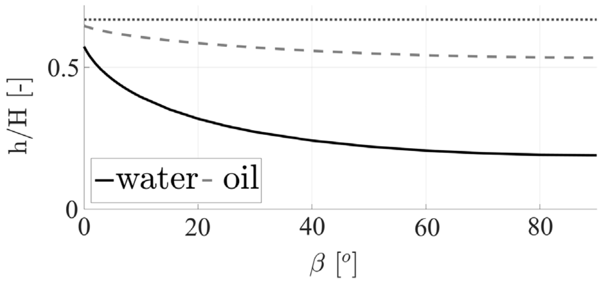
Appendix B. Two layered stratified flow

For two layered stratified flow ($h_2 = 0$), expressions of the wall friction at the lower wall f^l_w ($y = -h$), at the fluids interface f_I ($y = 0$) and at the upper wall f^u_w ($y = H - h$) become:

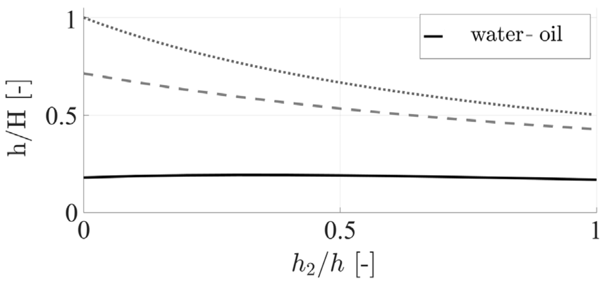
$$f^l_w = \frac{1}{(h\mu_G + (H-h)\mu_L)\rho_L\overline{U}_L^2}\Bigg(-h^2\mu_G\left(\frac{dp}{dx} - g\sin(\beta)\rho_L\right) + (h-H)\mu_L\left(\frac{dp}{dx}(h+H) + g(h-H)\sin(\beta)\rho_G - 2gh\sin(\beta)\rho_L\right)\Bigg), \tag{12a}$$

$$f_I = \frac{1}{(h\mu_G + (H-h)\mu_L)\rho_G\overline{U}_G^2}\Bigg(-(h-H)^2\mu_L\left(\frac{dp}{dx} - g\sin(\beta)\rho_G\right) + h^2\mu_G\left(\frac{dp}{dx} - g\sin(\beta)\rho_L\right)\Bigg), \tag{12b}$$

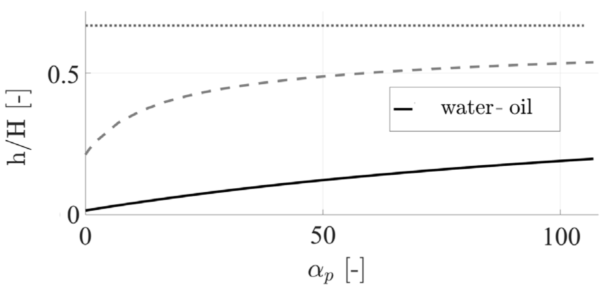
$$f^u_w = \frac{1}{(h\mu_G + (H-h)\mu_L)\rho_G\overline{U}_G^2}(h-H)^2\mu_L\left(\frac{dp}{dx} - g\sin(\beta)\rho_G\right)$$



(a) $h_2 = h/2$, $\frac{dp}{dx} = -20$ cPa.



(b) $\beta = 90°$, $\frac{dp}{dx} = -20$ cPa.



(c) $\beta = 90°$, $h_2 = h/2$.

Fig. 8. Illustration of normalised lower liquid layer thickness h/H associated with first (smallest h/H) local maxima of lower interfacial friction factor magnitude $|f_I^l|$ for stratified water/air (black full line) and oil/air (grey dashed line) flow for $H = 0.25$ cm and liquid layer thickness ratio $\alpha_2 = \frac{h_2}{h}$ as a function of: (a) Inclination angle β for $h_2 = h/2$ ($\alpha_2 = 0.5$) and $\frac{dp}{dx} = -20$ cPa, (b) Liquid layer thickness ratio α_2 for $\beta = 90°$ and $\frac{dp}{dx} = -20$ cPa, (c) α_p with $\alpha_p = (\frac{dp}{dx})/(\frac{dp}{dx})_0$, $\alpha_2 = 0.5$ and $\beta = 90°$. Normalised maximum lower liquid layer thickness within the channel yields $\frac{h}{H} = 1/(1+\alpha_2)$ (dotted line).

$$
\begin{aligned}
&- h\mu_G\left(\frac{dp}{dx}(h-2H)\right.\\
&\left.- 2g(h-H)\sin(\beta)\rho_G + gh\sin(\beta)\rho_L\right). \qquad (12c)
\end{aligned}
$$

For $h = 0$, f_I and f_w^l reduce to Eq. (10) as expected for a single fluid layer whereas f_w^u reduces to the negative of Eq. (10) so that $f_w^u = -f_w^l$ as expected for a single fluid layer and the magnitude corresponds to (10). Note that in general the lower and upper wall friction factor magnitude is no longer similar, *i.e.* $|f_w^l| \neq |f_w^u|$. Analytical expressions of the friction velocity (7) follow straightforwardly using (12).

Appendix C. Three layered stratified flow

For three layered stratified flow, expressions of the wall friction at the lower wall f_w^l $(y = -h)$, at the lower fluids interface f_I^l $(y = 0)$, at the upper fluids interface f_I^u $(y = H - h - h_2)$ and at the upper wall f_w^u $(y = H - h)$ become:

$$
\begin{aligned}
f_w^l =& \frac{1}{\rho_L \bar{U}_L^2\left(\mu_G(h+h_2) - \mu_L(h-H+h_2)\right)}\\
&\times \mu_G\left(\frac{dp}{dx}\left(-h^2 - 2Hh_2 + h_2^2\right)\right.\\
&\left.- 2gh_2\sin(\beta)\rho_G(h-H+h_2) + g\sin(\beta)(h+h_2)^2\rho_L\right)\\
&+ \mu_L(h-H+h_2)\left(\frac{dp}{dx}(h+H-h_2)\right.\\
&\left.+ g\sin(\beta)\rho_G(h-H+h_2) - 2gh\sin(\beta)\rho_L\right), \qquad (13a)
\end{aligned}
$$

$$
\begin{aligned}
f_I^l =& \frac{-1}{\rho_G \bar{U}_G^2\left(\mu_G(h+h_2) - \mu_L(h-H+h_2)\right)}\\
&\times \mu_L(h-H+h_2)^2\left(\frac{dp}{dx} - g\sin(\beta)\rho_G\right)\\
&- \mu_G\left(\frac{dp}{dx}\left(h^2 + 2hh_2 - 2Hh_2 + h_2^2\right)\right.\\
&\left.- 2gh_2\sin(\beta)\rho_G(h-H+h_2) - g\sin(\beta)\left(h^2 - h_2^2\right)\rho_L\right), \qquad (13b)
\end{aligned}
$$

$$
\begin{aligned}
f_I^u =& \frac{-1}{\rho_G \bar{U}_G^2\left(\mu_G(h+h_2) - \mu_L(h-H+h_2)\right)}\\
&\times \mu_G\left(\frac{dp}{dx}\left(h^2 - 2hH + 2hh_2 + h_2^2\right)\right.\\
&\left.- 2gh\sin(\beta)\rho_G(h-H+h_2) + g\sin(\beta)\left(h^2 - h_2^2\right)\rho_L\right)\\
&- \mu_L(h-H+h_2)^2\left(\frac{dp}{dx} - g\sin(\beta)\rho_G\right), \qquad (13c)
\end{aligned}
$$

$$
\begin{aligned}
f_w^u =& \frac{1}{\rho_L \bar{U}_{L_2}^2\left(\mu_L(h-H+h_2) - \mu_G(h+h_2)\right)}\\
&\times \frac{dp}{dx}(h-H+h_2)\left(2h\mu_G - \mu_L(h-H+h_2)\right)\\
&+ \frac{dp}{dx}\left(2h_2\mu_L(h-H+h_2) - \mu_G(h+h_2)^2\right)\\
&+ g\sin(\beta)\left(\mu_L(h-H+h_2)\left(\rho_G(h-H+h_2) - 2h_2\rho_L\right)\right.\\
&\left.+ \mu_G\left((h+h_2)^2\rho_L - 2h\rho_G(h-H+h_2)\right)\right). \qquad (13d)
\end{aligned}
$$

These expressions reduce for symmetrical liquid layers ($h_2 = h$) to:

$$ f_w^l = -\frac{\frac{dp}{dx}H + g(2h-H)\sin(\beta)\rho_G - 2gh\sin(\beta)\rho_L}{\rho_L \overline{U}_L^2}, \qquad (14a) $$

$$ f_I^l = \frac{(2h-H)\left(\frac{dp}{dx} - g\sin(\beta)\rho_G\right)}{\rho_G \overline{U}_G^2}, \qquad (14b) $$

$$ f_I^u = -f_I^l, \qquad (14c) $$

$$ f_w^u = -f_w^l. \qquad (14d) $$

It is easily seen that for $h_2 = h = 0$, (14) reduces to (10) as expected for a single fluid layer.

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